$use\ Z^n - 1 = 0\ to\ show\ that$

$$\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\left(\sin\frac{3\pi}{n}\right)\dots\left(\sin\frac{(n-1)\pi}{n}\right) = \frac{n}{2^{n-1}}$$

$(sol.)$

$$Z^n = 1 \quad \Rightarrow \quad Z = (1)^{\frac{1}{n}} \quad \Rightarrow \quad r = 1\,, \theta = 0$$

$$Z_k = \left(e^{\frac{i2k\pi}{n}}\right)$$

$$roots\ are:\ 1\ ,\ e^{\frac{i2\pi}{n}}\ ,\ e^{\frac{i4\pi}{n}}\ ,\ e^{\frac{i6\pi}{n}}\ ,\dots\dots\dots,\ e^{\frac{i2(n-1)\pi}{n}}$$

$$\because \prod_{k=0}^{n-1} Z_k = (-1)^{n-1}$$

$$\therefore e^{\frac{i2\pi}{n}} * e^{\frac{i4\pi}{n}} * e^{\frac{i6\pi}{n}} * \dots * e^{\frac{i2(n-1)\pi}{n}} = (-1)^{n-1}$$

$$\therefore e^{\frac{i\pi}{n}} * e^{\frac{i2\pi}{n}} * e^{\frac{i3\pi}{n}} * \dots * e^{\frac{i(n-1)\pi}{n}} = \sqrt{(-1)^{n-1}} = (i)^{n-1} \text{ **************** } (1)$$

$$\because \sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{e^{-ix}}{2i}\left(e^{i2x} - 1\right)$$

$$\therefore \sin\frac{\pi}{n} = \frac{e^{-\frac{i\pi}{n}}}{2i}\left(e^{i\frac{2\pi}{n}} - 1\right), \sin\frac{2\pi}{n} = \frac{e^{-\frac{i2\pi}{n}}}{2i}\left(e^{i\frac{4\pi}{n}} - 1\right), \sin\frac{(n-1)\pi}{n} = \frac{e^{-\frac{i(n-1)\pi}{n}}}{2i}\left(e^{i\frac{2(n-1)\pi}{n}} - 1\right)$$

$$L.H.S = \left(\sin\frac{\pi}{n}\right)\left(\sin\frac{2\pi}{n}\right)\left(\sin\frac{3\pi}{n}\right)\dots\left(\sin\frac{(n-1)\pi}{n}\right) =$$

$$\left(\frac{e^{-\frac{i\pi}{n}}}{2i}\left(e^{i\frac{2\pi}{n}} - 1\right)\right)\left(\frac{e^{-\frac{i2\pi}{n}}}{2i}\left(e^{i\frac{4\pi}{n}} - 1\right)\right)\left(\frac{e^{-\frac{i3\pi}{n}}}{2i}\left(e^{i\frac{6\pi}{n}} - 1\right)\right)\dots\left(\frac{e^{-\frac{i(n-1)\pi}{n}}}{2i}\left(e^{i\frac{2(n-1)\pi}{n}} - 1\right)\right) =$$

$$\frac{1}{(2i)^{n-1}} * \left(\frac{1}{e^{\frac{i\pi}{n}} * e^{\frac{i2\pi}{n}} * \dots * e^{\frac{i(n-1)\pi}{n}}}\right) * \left[\left(e^{i\frac{2\pi}{n}} - 1\right)\left(e^{i\frac{4\pi}{n}} - 1\right)\left(1 - e^{i\frac{6\pi}{n}}\right)\dots\left(e^{i\frac{2(n-1)\pi}{n}} - 1\right)\right] =$$

$$\frac{1}{2^{n-1}} * \frac{1}{i^{n-1}} * \frac{1}{i^{n-1}} * \left[\left(e^{i\frac{2\pi}{n}} - 1\right)\left(e^{i\frac{4\pi}{n}} - 1\right)\left(1 - e^{i\frac{6\pi}{n}}\right)\dots\left(e^{i\frac{2(n-1)\pi}{n}} - 1\right)\right] =$$

$$\frac{1}{2^{n-1}} * \frac{1}{(-1)^{n-1}} * \left[(-1)^{n-1}\left(1 - e^{i\frac{2\pi}{n}}\right)\left(1 - e^{i\frac{4\pi}{n}}\right)\left(1 - e^{i\frac{6\pi}{n}}\right)\dots\left(1 - e^{i\frac{2(n-1)\pi}{n}}\right)\right] =$$

$$\frac{1}{2^{n-1}} * \left[\left(1 - e^{i\frac{2\pi}{n}}\right)\left(1 - e^{i\frac{4\pi}{n}}\right)\left(1 - e^{i\frac{6\pi}{n}}\right)\dots\left(1 - e^{i\frac{2(n-1)\pi}{n}}\right)\right]$$

$$\therefore L.H.S = \frac{1}{2^{n-1}} * \left[\left(1 - e^{i\frac{2\pi}{n}}\right)\left(1 - e^{i\frac{4\pi}{n}}\right)\left(1 - e^{i\frac{6\pi}{n}}\right)\dots\left(1 - e^{i\frac{2(n-1)\pi}{n}}\right)\right] \text{ *************** } (2)$$

$$\because roots\ of\ (Z^n - 1)\ are:\ 1\ ,\ e^{\frac{i2\pi}{n}}\ ,\ e^{\frac{i4\pi}{n}}\ ,\ e^{\frac{i6\pi}{n}}\ ,\dots\dots\dots,\ e^{\frac{i2(n-1)\pi}{n}}$$

$$\therefore Z^n - 1 = (Z - 1)\left(Z - e^{i\frac{2\pi}{n}}\right)\left(Z - e^{i\frac{4\pi}{n}}\right)\left(Z - e^{i\frac{6\pi}{n}}\right)\dots\left(Z - e^{i\frac{2(n-1)\pi}{n}}\right)$$

$$\therefore \frac{Z^n - 1}{Z - 1} = \left(Z - e^{i\frac{2\pi}{n}}\right)\left(Z - e^{i\frac{4\pi}{n}}\right)\left(Z - e^{i\frac{6\pi}{n}}\right)\dots\left(Z - e^{i\frac{2(n-1)\pi}{n}}\right)$$

$substitute\ Z\ by\ 1\ to\ become\ like\ the\ unkown\ part\ of\ equation\ (2)$

$$\left(1 - e^{i\frac{2\pi}{n}}\right)\left(1 - e^{i\frac{4\pi}{n}}\right)\left(1 - e^{i\frac{6\pi}{n}}\right)\dots\left(1 - e^{i\frac{2(n-1)\pi}{n}}\right) = \lim_{Z\to 1}\frac{Z^n - 1}{Z - 1} = \frac{n}{1} * (1)^{n-1} = n$$

$$\therefore \left(1 - e^{i\frac{2\pi}{n}}\right)\left(1 - e^{i\frac{4\pi}{n}}\right)\left(1 - e^{i\frac{6\pi}{n}}\right)\dots\left(1 - e^{i\frac{2(n-1)\pi}{n}}\right) = n \quad \text{***** } substitute\ in\ equation\ (2)$$

$$L.H.S = \frac{n}{2^{n-1}} = R.H.S \quad \text{^\_^}$$